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RADIATIVE INSTABILITY OF STRATIFIED SHEAR FLOWS IN THE DRAZIN MODEL*

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The results of analytical stability calculations of an ideal liquid stratified shear flow described by the Drazin model /1/ are presented. It is shown that the previously found solution /1/, which gives the boundaries of the region of stability in a plane with parameters αJ, where α is a dimensionless wave number and J is the minimum value of the Richardson number are generally incomplete. The presence of constant stratification in the whole flow region enables free internal waves to exist which, emanating from the shear layer, give rise to additional radiative instability, previously examined in /2/ for a flow with tangential velocity discontinuity. The influence of the outer boundaries on the stability of the shear flow is clarified and the analytical results obtained are compared with numerical calculations carried out in /3, 4/ for certain models with smooth velocity and density profiles.

It was established in /3, 4/ that all modes involved in the shear flow can be divided into two classes. These are (in the terminology of /4/) A modes, exponentially decaying along the vertical z coordinate in both directions of the shear layer, and B modes which have a running wave structure along z (radiating from the shear layer). Both types of mode can give rise to instability with increments comparable in value, but over various ranges of wave number, and radiating instabilities linked to B modes are usually characterized by long wave-lengths. In the Drazin model examined below all possible regions of instability in the αJ plane, whether caused by A or B modes, are determined. The usual approach was used for this, comprising linearized hydrodynamics equations reducing to one singular Taylor-Goldstein equation (see below) which, together with the boundary conditions, form a boundary value problem in the eigenvalues of the harmonic-perturbation phase velocity c. The neutral stability curve was found from this problem which, in the plane with parameters αJ, divides regions with complex values demonstrating the instability of the flow (in view of the selfconjugacy of the boundary value problem, complex values of c may only appear as conjugate pairs in which one value corresponds to increasing and the other to decreasing perturbation), and regions with possible real values of c, corresponding to neutrally stable flows.

The analytically unstable solutions found in the form of B modes, caused by stratification, can be explained /5/ by the concept of a negative energy wave, which is typical for an unbalanced system; shear flow is one example of this. The gradual decrease in the negative energy of such waves due to the generation and propagation (radiation) of internal waves in a stratified liquid (or due to other energy selective factors such as viscous dissipation) leads to an increase in the absolute value of the energy, and therefore also of the wave amplitude, i.e. to instability.

1. Consider the plane-parallel motion of an ideal luquid along the horizontal x axis with velocity and density profiles in the form /1/

$$U(z) = U_0 \operatorname{th}(z/d);\ \rho(z) = \rho_0 e^{-\beta z} \tag{1.1}$$

The vertical perturbation component of the velocity $w(x,z,t) = W(z)e^{ik(x-ct)}$ described by the Taylor-Goldstein (TG) equation in the Boussinesq approximation /6-8/

$$W''(z) - \left[k^2 + \frac{U''}{U-c} - \frac{N^2}{(U-c)^2}\right] W(z) = 0 \tag{1.2}$$

where $N^2 = -g\rho'(z)\rho^{-1}(z)$ is the square of the buoyancy frequency (g is the acceleration due to gravity).

We will seek solutions of this equation with real values of k and c, lying on the neutral stability curve which divides the stability and instability regions in a plane with parameters $\alpha = kd$ (the dimensionless wave number) and $J = \min[N(z)/U'(z)]^2 = g\beta d^2/U_0^2$ (the minimum value of the Richardson number). Such solutions are conventionally called singular neutral modes.

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In addition, let us take the boundary conditions along z to be symmetrical (their specific form will be discussed below) which, in parallel with the symmetry along z of Eq.(1.2), allows the eigenvalue c to be set equal to zero for the required singular neutral mode (see /1/). This fundamental assumption $c = 0$ enables us to investigate the solutions of Eq.(1.2) analytically. As will become evident later, however, the phase velocity can be non-zero, even in a symmetrical problem, for radiative modes.

We will write Eq.(1.2) in dimensionless form, assuming $c = 0$, by introducing the typical scale of velocity U_0 and length d and using the function $u(z_*) = \operatorname{th}(z_*)$ as an independent variable.

$$\frac{d}{du}\left[(1-u^2)\frac{df}{du}\right] + \left[2 - \frac{\alpha^2 - J}{1-u^2} + \frac{J}{u^2}\right] f(u) = 0 \tag{1.3}$$

where $f(u) = f(\operatorname{th} z_*) = W(z)$, $z_* = z/d$ is the dimensionless vertical coordinate (henceforth we will omit the asterisk).

The equation derived possesses four regular singular points (where $u = 0, \pm 1, \infty$) and belongs to the Fuchs-type equation /9/. Let us examine the behaviour of the solution in the neighbourhood of the singular points.

Retaining the principal terms in (1.3), as $u \to 0$ we obtain the solution in the form

$$f = c_1 u^{\mu_1} + c_2 u^{\mu_2}, \quad \mu_{1,2} = {}^1/_2 \pm {}^1/_2 \sqrt{1-4J} \tag{1.4}$$

where $c_{1,2}$ are arbitrary constants. This solution approaches zero as $u \to 0$, i.e. also as $z \to 0$.

Close to the singular points $u = \pm 1$ (i.e. as $z \to \pm\infty$) Eq.(1.3) presents two different types of solution depending on the sign of the difference $J - \alpha^2$. If $J < \alpha^2$, then

$$f \sim (1 \pm u)^{\Lambda}, \quad \Lambda = {}^1/_2 \sqrt{\alpha^2 - J} \tag{1.5}$$

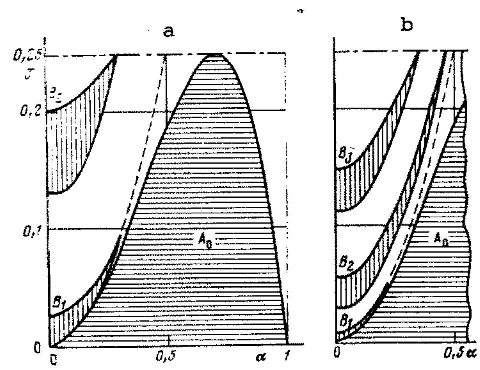
is a physically understandable solution as $u \to \pm 1$.

The second linearly independent solution $f \sim (1 \pm u)^{-\Lambda}$ increases without limit as $u \to \pm 1$.

The asymptotic solution (1.5) tends monotonically to zero as $u \to \pm 1$ $(z \to \pm\infty)$. This precise case was examined in /1/, where the expression for the neutral curve of the mode A: $J = \alpha^2(1-\alpha^2)$, localized along z was found. For $J > \alpha^2$, however, the asymptotic solution (1.3) as $u \to \pm 1$ has a different form:

$$f = c_3 \cos(\nu \ln \eta) + c_4 \sin(\nu \ln \eta), \tag{1.6}$$
$$\nu = {}^1/_2 \sqrt{J - \alpha^2}, \quad \eta = 1 - |u|$$

This solution does not vanish as $u \to \pm 1$ $(z \to \pm\infty)$ but has an oscillating character which corresponds to "radiative" boundary conditions, i.e. it describes the superposition of two modes travelling to the shear layer and from it to $z = \pm\infty$. The parabola $J = \alpha^2$ serves as the boundary in the αJ plane, which divides the A modes localized along z and the radiative B modes. The neutral stability curve, found in /1/ lies entirely under this parabola (see the figure).



Let us examine the region under the parabola, i.e. $J > \alpha^2$. We convert Eq.(1.3), separting out in explicit form the behaviour of the unknown function (see (1.4), (1.6)) in the neighbourhood of the singular points

$$f(u) = \xi^{1/4+1/2\gamma}(1-\xi)^{i\nu}\Phi(\xi),\ \ \xi = u^2,\ \ \gamma = {}^1/_2\sqrt{1-4J} \tag{1.7}$$

The new function $\Phi(\xi)$ satisfies the equation

$$\frac{d^2\Phi}{\partial\xi^2} + \frac{1+\gamma-(2+\gamma+2i\nu)\,\xi}{\xi\,(1-\xi)}\frac{d\Phi}{d\xi} - \left(i\nu+\frac{\gamma}{2}+\frac{5}{4}\right)\left(i\nu+\frac{\gamma}{2}-\frac{1}{4}\right)\frac{\Phi}{\xi\,(1-\xi)} = 0 \tag{1.8}$$

which differs somewhat from the hypergeometric equation /10/ usually examined, in that it contains the imaginary parameter $i\nu$. The number of singular points in Eq.(1.8) was reduced to three $(\xi = 0,\ 1,\ \infty)$, but a branching point $\xi = 0$ appeared.

We make a branch cut along the section $[0, 1]$. Its right edge corresponds to a change u from 0 to 1 and its left edge to a change in u from 0 to -1. At the transition from the right to the left edge, the solution of Eq.(1.8) will have a certain phase jump.

If $\gamma \neq 0$, the required solution has the form /9, 10/

$$\begin{aligned}
&\Phi = c_1\Phi_1{}^\circ(u) + c_2\Phi_2{}^\circ(u)\\
&\Phi_1{}^\circ(u) = F\left(i\nu+\frac{\gamma}{2}+\frac{5}{4},\ i\nu+\frac{\gamma}{2}-\frac{1}{4},\ \gamma+1, u^2\right)\\
&\Phi_2{}^\circ(u) = u^{-2\gamma}F\left(i\nu-\frac{\gamma}{2}+\frac{5}{4},\ i\nu-\frac{\gamma}{2}-\frac{1}{4},\ -\gamma+1, u^2\right)
\end{aligned} \tag{1.9}$$

Here $F(a, b, c, u^2)$ is the hypergeometric function containing complex parameters. The chosen form of the solution (1.9) is convenient for examining the asymptotic form close to $u = 0$ (at which the upper index $\Phi_{1,2}{}^\circ$ is indicated), since, as $u \to 0$ we have $F(a, b, c, u^2) \to 1$. To investigate the solution of Eq.(1.8) as $\xi\ (=u^2) \to 1$, it is convenient to write these solutions in terms of other functions

$$\begin{aligned}
&\Phi_1{}^1(u) = F\left(i\nu+\frac{\gamma}{2}+\frac{5}{4},\ i\nu+\frac{\gamma}{2}-\frac{1}{4},\ 2i\nu+1,\ 1-u^2\right)\\
&\Phi_2{}^1(u) = (1-u^2)^{-2i\nu}F\left(-i\nu+\frac{\gamma}{2}-\frac{1}{4},\ -i\nu+\frac{\gamma}{2}+\frac{5}{4},\ -2i\nu+1,\ 1-u^2\right)
\end{aligned} \tag{1.10}$$

which are connected with the functions $\Phi_{1,2}{}^\circ$ by the linear relationship /10/

$$\begin{aligned}
&\Phi_1{}^\circ = A_1{}^-\Phi_1{}^1 + A_1{}^+\Phi_2{}^1,\ \ \Phi_2{}^\circ = A_2{}^-\Phi_1{}^1 + A_2{}^+\Phi_2{}^1\\
&A_1{}^\pm = A_1{}^\pm(\gamma) = \Gamma(1+\gamma)\,\Gamma(\pm 2i\nu)\left[\Gamma\left(\frac{\gamma}{2}\pm i\nu-\frac{1}{4}\right)\times \Gamma\left(\frac{\gamma}{2}\pm i\nu+\frac{5}{4}\right)\right]^{-1}\\
&A_2{}^\pm = A_1{}^\pm(-\gamma)
\end{aligned} \tag{1.11}$$

We note that, as follows from the explicit form of the function $\Phi_2{}^\circ(u)$, for a transition though zero from positive to negative values of u, an additional factor $e^{2i\pi\gamma}$ appears in the function i.e. the phase changes in a jump by $2\pi\gamma$. In order to take this into account in the second relation (1.11) the coefficients $A_2{}^\pm$ must be multiplied by $e^{2i\pi\gamma}$ in the region of negative values of u $(-1 < u < 0)$. Thus, having the solution of the hypergeometric Eq.(1.8) for Φ and substituting it into (1.7), we obtain

$$f(u) = u^{1/2+\gamma}(1-u^2)^{i\nu}\,[c_1\Phi_1{}^\circ(u) + c_2\Phi_2{}^\circ(u)] = u^{1/2+\gamma}(1-u^2)^{i\nu}\times \begin{cases} [(c_1A_1{}^- + c_2A_2{}^-)\,\Phi_1{}^1(u) + (c_1A_1{}^+ + c_2A_2{}^+)\,\Phi_2{}^1(u)], & 0<u<1\\ [(c_1A_1{}^- + c_2A_2{}^-e^{2i\pi\gamma})\,\Phi_1{}^1(u) + (c_1A_1{}^+ + c_2A_2{}^+e^{2i\pi\gamma})\,\Phi_2{}^1(u)], & -1<u<0 \end{cases} \tag{1.12}$$

Solution (1.12) after changing to the initial variables may be considered as the accurate solution of the TG Eq.(1.2). We recall that this solution holds when $J > \alpha^2$ and, unlike the solution previously obtained /1/, it does not decrease but oscillates as $z \to \pm\infty$. Indeed, from (1.12) it follows that

$$\begin{aligned}
&W(z) = f(\operatorname{th} z) \sim (c_1A_1{}^- + c_2A_2{}^-)\,e^{2i\nu(\ln 2-z)} + (c_1A_1{}^+ + c_2A_2{}^+)\,e^{-2i\nu(\ln 2-z)},\quad z\to\infty\\
&W(z) \sim ie^{-i\pi\gamma}\,[(c_1A_1{}^- + c_2A_2{}^-e^{-2i\pi\gamma})\,e^{2i\nu(\ln 2+z)} + (c_1A_1{}^+ + c_2A_2{}^+e^{2i\pi\gamma})\,e^{-2i\nu(\ln 2+z)}],\quad z\to-\infty
\end{aligned} \tag{1.13}$$

2. On the basis of the analytical solutions of the TG equations obtained in Sect.1, let us determine the influence of solid boundaries in the model under examination /1/ on the formation and transformation of new regions of instability in the αJ plane, corresponding to radiative B modes. Let us assume that the flow is enclosed between solid impermeable plane boundaries $z = \pm h$. This corresponds to the zero boundary condition $f|_{z=\pm h} = 0$. We consider the dimensionless distance between the boundaries to be fairly large $(h \gg 1)$, for the asymptotic solution (1.13), to which zero boundary conditions are applied, to be sufficiently accurate in the neighbourhood of $z = \pm h$. This leads to a set of two uniform equations with respect to c_1, c_2, from the condition for the non-trivial solvability of which

$$A_1^- A_2^- e^{4i\nu(\ln 2 - h)} + A_1^+ A_2^+ e^{-4i\nu(\ln 2 - h)} + A_1^- A_2^+ + A_1^+ A_2^- = 0 \tag{2.1}$$

for a fixed value of J, we can find the permitted values of α for which, by our assumption, $c = 0$.

Expressions (1.11) for the coefficients A_1^- and A_1^+ (in a similar manner for A_2^- and A_2^-) differ only in the sign of the imaginary part of the arguments of the Γ-function. Hence it follows that /11/ A_1^- and A_1^+ (in the same way as A_2^- and A_2^+) have identical moduli and opposite phases. Therefore all products $A_1^+ A_2^{\pm}$ occurring in (2.1) have identical moduli. Thus (2.1) can be reduced to a simple trigonometric equation

$$\cos[\varphi_1 + \varphi_2 + 4\nu(\ln 2 - h)] + \cos(\varphi_1 - \varphi_2) = 0$$

with respect to the phases $\varphi_{1,2} = \varphi_{1,2}(\alpha, J)$ of the complex coefficients $A_{1,2}^-$.

This equation is satisfied for

$$\varphi_j + 2\nu(\ln 2 - h) = \pi/2 + n\pi, \; n = 0, \pm 1, \pm 2, \ldots \tag{2.2}$$

where either $j = 1$ or $j = 2$. Relations (2.2) determine the upper branch $(j = 1)$ and the lower branch $(j = 2)$ of the neutral stability curve in the αJ plane. It turns out that for a specified value of the dimensionless distance $2h$ between the boundaries, relations (2.2) are only solvable for a finite number of values of n which, in turn, give a finite number of instability zones, each of which correspond to a specific B mode.

The calculations carried out have demonstrated that the number m of instability zones increases as h increases as given by $m \approx 0{,}16h$. The boundaries of the instability zone are shown in the figure for $h = 10$ (a) and $h = 20$ (b) and fully coincide with those derived /3/ by the direct numerical method of solving TG Eqs.(1.2) with corresponding boundary conditions. The curve $J = \alpha^2$ is shown by the broken line, which divides the regions in which exponential A modes (the horizontal dashed line) and radiative B modes (the vertical dashed line) exist. The level $J = 1/4$ is marked by the dash-dot line above which, in accordance with the Miles-Howard theorem /7, 8/ instability is not possible.

Let us make two observations about the characteristic features of B type instability zones.

Firstly, as calculations by relation (2.2) show and as is evident from the figure, there are zones (B_2, B_3) removed from the common boundary $J = \alpha^2$ for each radiative region and zones pressed against this boundary (for the case depicted in the figure with $h = 10, 20$ there is only one such zone B_1). The remote zones B_2 and B_3 typically have an upper and lower boundary which move away upwards and close up exactly at the level $J = 1/4$. This is clear from the two determined boundaries of the zones for Eqs.(2.2) which coincide where $J = 1/4\,(\nu = 0)$, i.e. they share a common point in the αJ plane since the complex coefficients A_1^- and A_2^- coincide under these conditions and, of course, the phases φ_1 and φ_2 also. For the compressed zone B_1 its upper boundary does not manage to reach the level $J = 1/4$ and intersects the parabola $J = \alpha^2\,(\nu = 0)$, which obeys Eq.(2.2), i.e. it forms the lower boundary for the instability zones pressed against it.

As h increases, so the number of instability zones increases, but they become narrower, shift to the right and come up against the parabola $J = \alpha^2$ (this tendency can be seen in the figure). Their maximum shifts downwards along the parabola $J = \alpha^2$. Calculations show that, for $h \approx 100$, all zones run adjacent to the parabola and their maxima are significantly less then $1/4$. Since they are immediately adjacent to the Drazin zone A_0 in the region of small α and J, these zones transform zone A in this region of parameters. In the limit as $h \to \infty$ all instability zones B_j gather together at point $\alpha = J = 0$ (see on these lines Sect.4) and an unperturbed boundary of zone A_0 remains in the plane, first calculated by Drazin /1/. All the special features noted are confirmed by numerical solutions of the TG equation /3, 4/.

Secondly, we note that the presence of a discreet spectrum of modes (B_j, A_j), is connected with the solid boundaries which form a waveguide for internal waves on the shear flow. Instability regions under the curve $J = \alpha^2$ correspond to exponentially decaying A modes and instability regions above this curve are related to oscillating B modes. The division of modes into exponential and oscillating, however, only makes sense for large distances between the boundaries $(h \gg 1)$. If $h \leqslant 1$, then a common region of instability is formed, which was obtained numerically /3/ (see also /7/).

3. Now let us examine a semi-enclosed model, containing a solid boundary for $z=-h$ and unlimited in height (a model of the earth's atmosphere with a non-uniform wind). Once again let us examine the case $h \gg 1$ using the asymptotic forms (1.13). The boundary condition for $z=-h$ remains as before, but as $z \to \infty$ we apply a radiation condition, i.e. we consider that the solution takes the form of a wave travelling from the shear layer (the wave is taken to be travelling towards $z=\infty$ if its group velocity is in the direction of increasing z). This condition means that one of the coefficients in (1.13), corresponding to the wave which arrives from infinity, should be net equal to zero. Thus we derive a uniform set of equations in c_1 and c_2, on equating the determinant of which to zero leads to the relation

$$2i \sin(\pi\gamma) A_1^- A_2^- e^{i\pi\gamma} = (A_1^+ A_2^- - A_1^- A_2^+ e^{2i\pi\gamma}) e^{-4i\nu(\ln 2 - h)} \tag{3.1}$$

Taking into account the equality of the moduli and the opposite signs of the phase of the coefficients A_1^- and A_1^+ and also A_2^- and A_2^+, we conclude that relation (3.1) holds when the equality

$$\varphi_1 + \varphi_2 + 4\nu(\ln 2 - h) - 2k\pi = 0 \tag{3.2}$$

is simultaneously satisfied, which guarantees the equality of the arguments (to within $2\pi k$) of the two terms in Eq. (3.1) and one from the equalities

$$\begin{gathered} \varphi_1 - \varphi_3 + \pi(2m+1) = 0 \\ \varphi_1 - \varphi_3 + 2\pi(n+\gamma) = 0 \end{gathered} \tag{3.3}$$

which give equality of the moduli of the terms in the same initial Eq. (3.1). In relations (3.2) and (3.3), k, m, n are integers.

The general solution of Eqs. (3.2) and (3.3) for fixed k, m, n and h defines two points in all in the αJ plane. By varying k, m, n for fixed h it is possible to obtain several pairs of points to which various eigenmodes apply.

Thus, for a semi-enclosed model, instead of a neutral stability curve with $\operatorname{Re} c = 0$ in the αJ plane only separate points are obtained. This is explained by the fact that in the given situation, due to the asymmetry of the model, the value of $\operatorname{Re} c$ along the neutral curve does not remain constant or equal to zero but is always changing. If the axis $\operatorname{Re} c$ is introduced above the plane αJ the neutral instability curve will lie in the three-dimensional space already formed and its intersection with the αJ plane will define two points corresponding to solutions (3.2) and (3.3), which are only valid when $\operatorname{Re} c = 0$. These results agree well with direct numerical calculations /4/.

4. For a model unbounded in both directions the general solution (1.12) of the TG equation must satisfy the radiation conditions as $z \to \pm\infty$. This signifies that its asymptotic representation (1.13) must only contain those exponents which correspond to waves travelling along z from the shear layer. By equating the coefficients of the "spere" exponents to zero, we obtain a uniform set of equations in c_1, c_2 and by equating to zero the determinant of this we get

$$A_1^+ A_2^+ \sin \pi\gamma = 0 \tag{4.1}$$

From the form of the coefficients $A_{1,2}^{\pm}$ it follows that Eq. (4.1) can be satisfied where $\gamma \neq 0$ (i.e. $J \neq 1/4$) only in the case, where $\alpha = J = 0$, which agrees with the limit transition discussed in Sect. 2. This result does not mean, however, that in the unbounded model only the A_0 mode is growing: other modes can also be growing (see, as an example, /12/), but for these $\operatorname{Re} c \neq 0$ and, therefore, the analytical approach developed is inapplicable to them.

5. The analysis of the Drazin model for three different cases (with two outer boundaries, semi-enclosed and unbounded) demonstrates that in shear flow, together with exponentially decaying modes localized in the neighbourhood of the shear layer, radiative modes may also exist if the liquid is stratified a long way from the shear layer. This conclusion is supported by the results obtained in /13, 14/, in which a Holmbow model was studied with the same velocity profile as in the case of the Drazin model, but with a different density profile representing a smooth transition between two costant density values for $z = \pm\infty$. As these papers show, there are no other modes in the Holmbow model apart from exponentially decaying ones, since the liquid is uniform at infinity.

In cases where there is stratification away from the shear layer, it is essential to consider radiative modes, for this will lead to the appearance of additional instability zones, whose number and relative positions depend on the presence of one or both outer boundaries and also on the distance between these boundaries (or between one of them and the shear layer).

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REFERENCES

1. DRAZIN P.G., The stability of a shear layer in an unbounded heterogenous inviscid fluid. J. Fluid Mech, 4, 2, 1958.
2. OSTROVSKII L.A. and TSIMRING L.SH., Radiative instability of shear flow in a stratified liquid. Izv. Akad. Nauk SSSR, Fiz. Atmosfery i Okeana, 17, 7, 1981.
3. EINAUDI F. and LALAS D.P., The effect of boundaries on the stability of inviscid stratified shear flows. Trans. ASME. Ser E. J. Appl. Mech. 43, 2, 1976.
4. FRITTS D.C., Simple stability limits for vertically propagating unstable modes in a $\tanh(z)$ velocity profile with a rigid lower boundary. J. Atmos. Sci. 37, 7, 1980.
5. OSTROVSKII L.A., STEPANYANTS U.A.and TSIMRING L.SH.,The interaction of internal waves with flows and turbulence in the ocean. Non-linear Waves. Selforganization. Edited by A.V. Gapanov-Grekhov and M.I. Rabinovich. Moscow, Nauka, 1983.
6. DRAZIN P.G. and HOWARD L.N., Hydrodynamic stability of parallel flow of inviscid fluid. Advances in Applied Mechanics. N.Y., Acad. Press, 9, 1966.
7. TURNER J., Buoyancy Effects in Fluid /Russian translation/, Moscow, Mir, 1977.
8. GOSSARD E. and COOK W., Waves in the Atmosphere, /Russian translation/, Mir, Moscow, 1978.
9. KPATSER A. and FRANZ V., Transcendental Functions, /Russian translation/, Izd-vo insotr. lit. Moscow, 1963.
10. BATEMAN H. and ERDELYI A., Higher Transcendental Functions, /Russian translation/, 1, Nauka, Moscow, 1973.
11. YANKE E., EMDE F. and LOSCH F., Special Functions, /Russian translation/, Nauka, Moscow, 1968.
12. DRAZIN P.G., ZATURSKA M.B. and BANKS H.H., On the normal modes of parallel flow of inviscid stratified fluid. Pt 2. Unbounded flow with propagation at infinity. J. Fluid Mech. 95, 4, 1979.
13. MILES J.W., On the stability of heterogeneous shear flows. Pt.2 J. Fluid Mech. 16, 2, 1963.
14. ROMANOVA N.N. and TSEITLIN V.U., Evolution of a non-linear wave package against the background of losing stability of a stratified shear flow. Izv. Akad. Nauk SSSR, Fiz Atmos. Okeana, 19, 8, 1983.

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NON-LOCAL NON-LINEAR EQUATIONS OF WIND WAVES OVER AN UNEVEN BOTTOM*

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The evolution of a two-layer water-air medium under the action of a wind is treated in the weak non-linearity approximation. Here, together with the effects studied in /1-3/, we present, using an operator method /4, 5/, analogies of the Boussinesq equations without any assumption regarding the shallowness of the water reservoir and also taking account of the action of a wind but under the assumption that the amplitudes of the corresponding wave processes are small and the average velocity of the wind and the bottom of the reservoir are specified functions which vary "slowly" with the horizontal coordinates and time. Non-local (pseudo-differential) equations are obtained which describe the behaviour of the medium being studied taking account of the quadratic and cubic non-linear terms. Asymptotic solutions of these equations which take account of weak resonance interactions are constructed using the methods in /6, 7/. Algorithms are given for deriving the analogous equations and the construction of their asymptotic solutions when account is taken of an arbitrary degree of non-linearity.

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